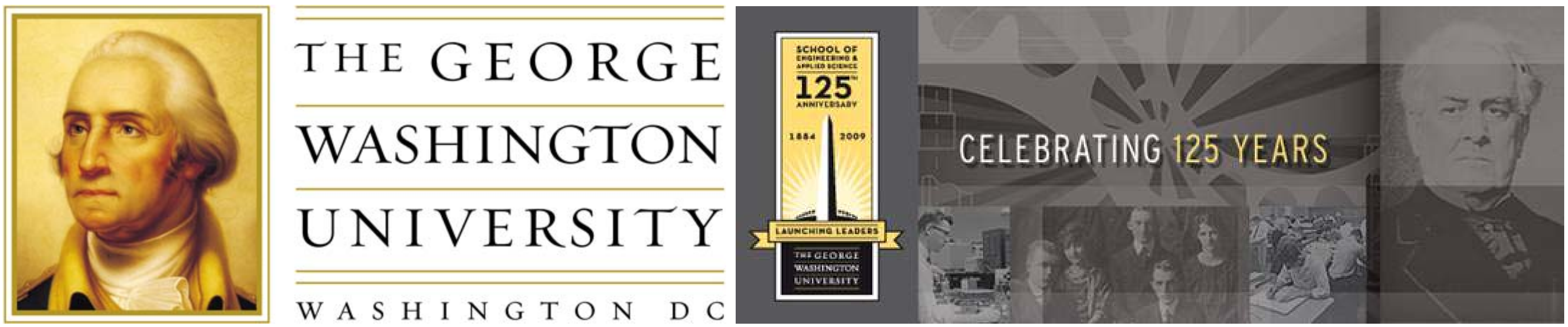
Three-Point Lifetime Distribution Elicitation for Maintenance Optimization in a Bayesian Context

"The State of the Art in the Use of Expert Judgment in Risk and Decision Analyses"

July 5th, 2017, *Delft University of Technology*



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2. THREE POINT ELICITATION
3. PRIOR DIRICHLET PROCESS CONSTRUCTION
4. BAYESIAN UPDATING USING FAILURE AND MAINTENANCE DATA
5. MAINTENANCE OPTIMIZATION
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1. INTRODUCTION

Problem Description...

- **Maintenance optimization** has been a focus of research interest.

- **Dekker (1996) and Mazzuchi *et al.* (2014)** provide an elaborate review and analysis of applications of maintenance optimization models.

> *"Besides, many textbooks on operations research use replacement models as examples"*, Dekker (1996).

- **A main bottleneck** in the implementation of maintenance optimization procedures is **the determination of the life length distributions**.

- Due to **scarcity of good component failure data**, determination via known statistical estimation procedures is, in many cases, impossible. **Why is that?**

1. INTRODUCTION

Problem Description...

Answer:

- Scarcity of failure data is **inherent to an efficient preventive maintenance environment.** The complete component life cycle will rarely be observed.

- **Occurrence of many failures,** on the other hand, will lead to **equipment modification**, making past data obsolete.

Proposed Solution:

- One approach to overcome this scarcity of data is to determine the lifetime distribution based on the **use of expert judgment.**

- In the absence of data, **normative experts** are tasked with **specifying distributions that are consistent** with a **substantive expert's** judgment, whom **may not be statistically trained**.

1. INTRODUCTION

Literature Review...

- **To facilitate** such a situation, **integration of graphically interactive and statistical elicitation procedures** for distribution modeling has been a topic of research for **quite some time** with **some re-invigoration more recently**.

- See, **DeBrota *et al.* (1989), Van Dorp (1989), AbouRizk *et al.* (1992), Van Noortwijk *et al.* (1992), Wagner and Wilson (1996).**

- **More recently : Van Dorp and Mazzuchi (2000), Garthwaite, Kadane and O'Hagan (2005) and Morris *et al.* (2014),** the latter developing a web-based distribution elicitation tool called 'MATCH', and **Shih N (2015).**

- Most of these indirect elicitation procedures "fit" continuous distribution to the elicited expert judgement, **but do not match the expert judgement exactly**, with the exception of **Van Dorp and Mazzuchi (2000) and Shih N (2015) who match two elicited quantiles uniquely to a beta distribution.**

GTSP Distribution...

- Herein, the elicitation of **lower and upper quantile estimates x_p and x_r** and **the most likely estimate η, $x_p < \eta < x_r$**, of a five-parameter **Generalized Two-Sided Power (GTSP) distribution** (Herrerías *et al.*, 2009) is proposed.

- The GTSP distribution with support (a, b) has prob. density function (pdf)

$$f(x|\Theta) = \mathcal{C}(\Theta) \times \begin{cases} \left(\frac{x-a}{\eta-a}\right)^{m-1}, & \text{for } a < x < \eta \\ \left(\frac{b-x}{b-\eta}\right)^{n-1}, & \text{for } \eta \leq x < b, \end{cases} \qquad (1)$$

where $\Theta = \{a, \eta, b, m, n)$ and

$$\mathcal{C}(\Theta) = \frac{mn}{(\eta - a)n + (b - \eta)m}. \qquad (2)$$

- The GTSP distribution was suggested as **a more flexible alternative to the classical beta distribution in the unimodal domain**.

1. INTRODUCTION

MR diagram GTSP Distribution…

- Moment Ratio (MR) diagrams plot kurtosis β_2 against $\sqrt{|\beta_1|}$ with convention that $\sqrt{|\beta_1|}$ retains the sign of skewness β_1.

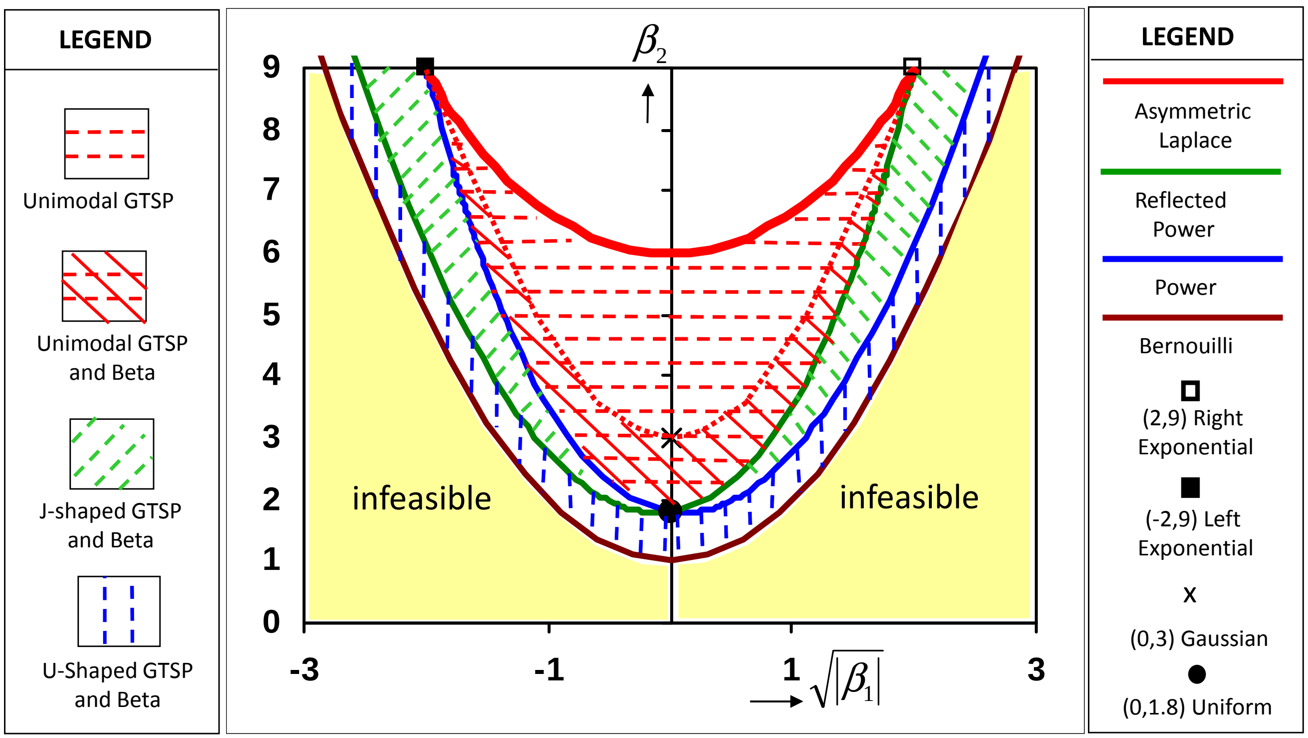

Figure 1. Moment Ratio ($\sqrt{|\beta_1|}$, β_2) coverage diagram for GTSP (1) and beta pdfs.

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Matching...

- **Given a fixed support (a, b)**, **chosen arbritrarily large**, standardize lower and upper quantile estimates x_p, x_r and most like value estimate η values **to values y_p, y_r and θ in $(0, 1)$** using **transformation $(x - a)/(b - a)$**.

- Utilizing that same linear transformation, the pdf (1) reduces to

$$f(y|m,n,\theta) = \frac{mn}{(1-\theta)m + \theta n} \times \begin{cases} \left(\frac{y}{\theta}\right)^{m-1}, & \text{for } 0 < y < \theta \\ \left(\frac{1-y}{1-\theta}\right)^{n-1}, & \text{for } \theta \leq y < 1. \end{cases} \quad (3)$$

$0 < \theta < 1, n, m > 0.$

- While the most likely value θ is elicited directly, the quantile estimates y_p, y_r are needed to **indirectly elicit the power-parameters m and n** of the pdf (3), hence the requirement $0 < y_p < \theta < y_r < 1$.

Matching...

- From pdf (3) one directly obtains **the cumulative distribution function:**

$$F(y|\Theta) = \begin{cases} \pi(\theta, m, n)\left(\frac{y}{\theta}\right)^m, & \text{for } 0 \le y < \theta \\ 1 - [1 - \pi(\theta, m, n)]\left(\frac{1-y}{1-\theta}\right)^n, & \text{for } \theta \le y \le 1, \end{cases} \quad (4)$$

with mode (or anti-mode) probability $\boldsymbol{Pr(X \le \theta) = \pi(\theta, m, n) = \theta n/[(1-\theta)m + \theta n]}$**.**

- **Given the quantile estimates** $\boldsymbol{y_p, y_r}$, **the quantile constraints below** need to be solved to obtain **the power-parameters** $\boldsymbol{m}$ **and** $\boldsymbol{n}$ in (3), (4):

$$\begin{cases} F(y_p|\theta, m, n) = \pi(\theta, m, n)\left(\frac{y_p}{\theta}\right)^m = p, \\ F(y_r|\theta, m, n) = 1 - [1 - \pi(\theta, m, n)]\left(\frac{1-y_r}{1-\theta}\right)^n = r. \end{cases} \quad (5)$$

Matching...

- It is proven that **the lower quantile constraint** in (5) defines **a unique implicit function** $m^{\bullet} = \xi(n)$, where $\xi(\cdot)$ is a **strictly increasing continuous concave function** in n, such that $\xi(n) \downarrow 0$ as $n \downarrow 0$ and $(m^{\bullet} = \xi(n), n)$ satisfies the first quantile constraint in (5) for all $n > 0$.

- As a result, when $n \downarrow 0$ the GTSP density $\boldsymbol{f(y|\xi(n), n, \theta)}$ converges to a Bernoulli distribution with **probability mass** $\boldsymbol{p}$ **at** $\boldsymbol{y = 0}$ and **probability mass** $\boldsymbol{1 - p}$ **at** $\boldsymbol{y = 1}$.

- Finally, it is proven that the implicit function $\xi(n)$ has **the following tangent line at** $\boldsymbol{n = 0}$:

$$M(n|p, \theta) = n \times \frac{\theta}{1-\theta} \times \frac{1-p}{p}, \tag{6}$$

where **in addition** for all values of $n > 0$, $\boldsymbol{M(n|p, \theta) \geq \xi(n)}$.

Matching...

- It is proven that **the upper quantile constraint** in (5) defines **a unique implicit function** $n^{\bullet} = \zeta(m)$, where $\zeta(\,\cdot\,)$ is a **strictly increasing continuous concave function** in m, such that $\zeta(m) \downarrow 0$ as $m \downarrow 0$ and $(m, n^{\bullet} = \zeta(m))$ satisfies the second quantile constraint in (5) for all $m > 0$.

- As a result, when $m \downarrow 0$ the GTSP density $\boldsymbol{f(y|m,\zeta(m),\theta)}$ converges to a Bernoulli distribution with **probability mass $\boldsymbol{r}$ at $\boldsymbol{y = 0}$** and **probability mass $\boldsymbol{1 - r}$ at $\boldsymbol{y = 1}$**.

- Finally, it is proven that the implicit function $\zeta(m)$ has **the following tangent line at $\boldsymbol{m = 0}$** :

$$N(m|r,\theta) = m \times \frac{1-\theta}{\theta} \times \frac{r}{1-r}, \tag{7}$$

where for all values of $\boldsymbol{m > 0,\ N(m|r,\theta) \geq \zeta(m).}$

Numerical Algorithm...

- From these conditions it follows that **the quantile constraint set (7)** has **a unique solution (m^*, n^*) where $m^*, n^* > 0$.**

- **The unique solution $m^\bullet = \xi(n)$ for a fixed value $n > 0$** may be solved using, e.g., GoalSeek in Microsoft Excel. **The unique solution $n^\bullet = \zeta(m)$ may be solved for a fixed value of $m > 0$** in a similar manner.

- **The following algorithm** now solves for **(m^*, n^*) where $m^*, n^* > 0$.**

Step 1: Set $n^\bullet = \delta > 0$ (arbitrarily small).
Step 2: Calculate $m^\bullet = \xi(n^\bullet)$ (satisfying first quantile constraint in (5)).
Step 3: Calculate $n^\bullet = \zeta(m^\bullet)$ (satisfying second quantile constraint in (5)).
Step 4: If $\left|\pi(\theta, m^\bullet, n^\bullet)\left(\frac{y_p}{\theta}\right)^{m^\bullet} - p\right| < \epsilon$ **Then** Stop **Else Goto** Step 2.

Example...

$$y_p = 1/6, \theta = 4/15, y_r = 1/2,\ p = 0.2,\ r = 0.8\ \Rightarrow$$
$$m^* \approx 1.506 \text{ and } n^* \approx 2.839.$$

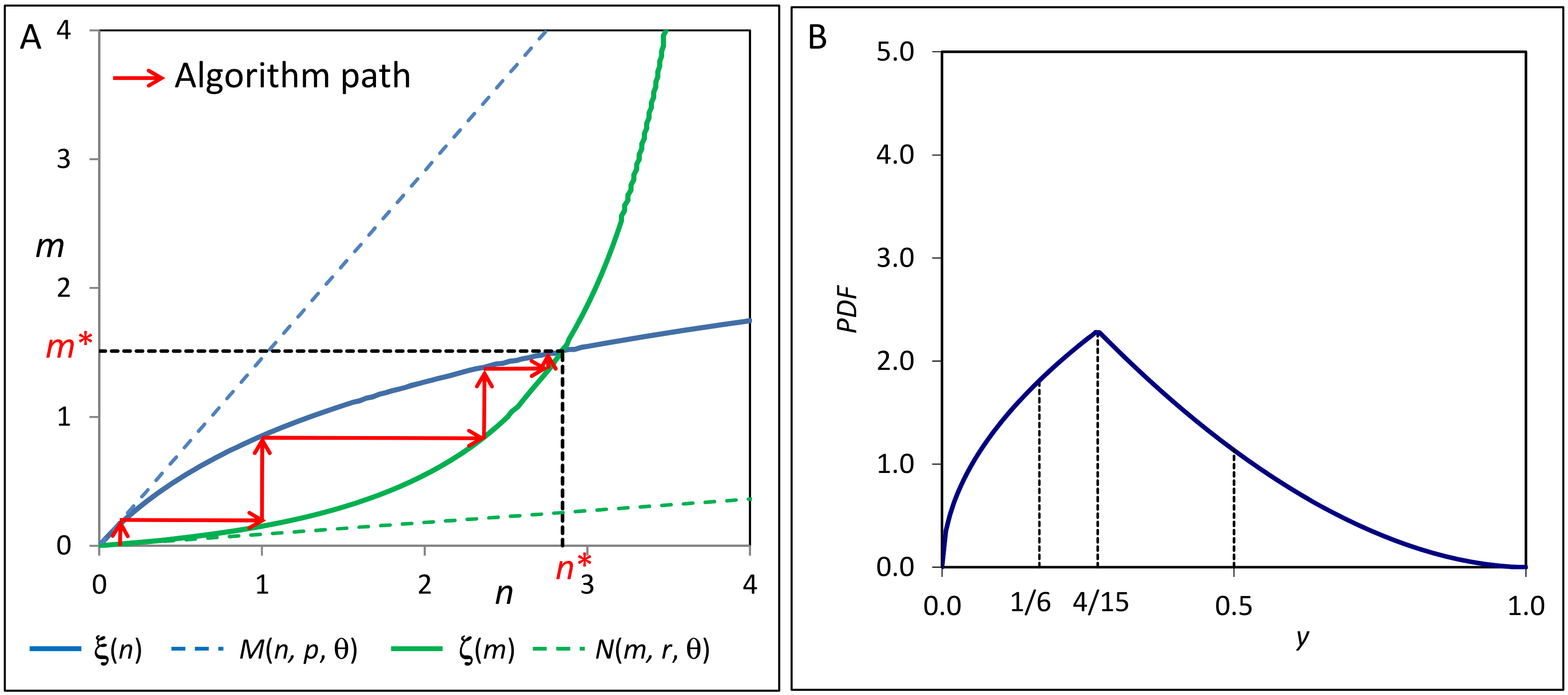

Figure 2. A: Implicit functions $\xi(n)$ and $\zeta(m)$ and algorithm path for the example data above B: GTSP pdf solution (11).

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3. PRIOR DIRICHLET PROCESS

Construction...

- **Aim:** Use elicited expert life time distributions $F_e(x), e = 1, \ldots, E$ to **specify the prior parameters of a Dirichlet Process**. **A Dirichlet process (Ferguson, 1973)** may be used to define a distribution for the cdf $F(x)$ for every time $x \in (0, \infty) = \mathbb{R}^+$. **Below a 5 step procedure is demonstrated.**

- **Ferguson (1973)** showed that for a DP with parameter measure $\alpha(\mathcal{A}) > 0$, $\mathcal{A} \subset \mathbb{R}^+$, $\boldsymbol{F(x) \sim Beta(\ \alpha\{(0,x)\},\ \alpha\{[x,\infty)\})}$. Thus with

$$\boldsymbol{\alpha(\mathbb{R}^+)} = \boldsymbol{\alpha\{(0,x)\}} + \boldsymbol{\alpha\{[x,\infty)\}}$$

we have

$$E[F(x)|\alpha(\,\cdot\,)] = \frac{\boldsymbol{\alpha\{(0,x)\}}}{\boldsymbol{\alpha(\mathbb{R}^+)}},$$

$$V[F(x)|\alpha(\,\cdot\,)] = \frac{\boldsymbol{\alpha\{(0,x)\}} \times \{\boldsymbol{\alpha(\mathbb{R}^+)} - \boldsymbol{\alpha\{(0,x)\}}\}}{\{\boldsymbol{\alpha(\mathbb{R}^+)}\}^2\{\boldsymbol{\alpha(\mathbb{R}^+)} + 1\}}.$$

3. PRIOR DIRICHLET PROCESS

Construction...

- **Step 1:** Set $F_d(x) = \frac{1}{E}\sum_{e=1}^{E} F_e(x) = \overline{F(x)}$ using **an equal-weighted linear opinion (see, e.g. Cooke, 1991)** since in Bayesian context data, hopefully, eventually outweighs the prior expert information.

Table 1. Illustrative example A: Support $[0, 30]$ B: Support $[0, 100]$

A	EXPERT 1	EXPERT 2	EXPERT 3		EXPERT 1	EXPERT 2	EXPERT 3
a	0	0	0	**a**	0	0	0
p	0.2	0.2	0.2	**p**	0.2	0.2	0.2
r	0.8	0.8	0.8	**r**	0.8	0.8	0.8
b	**30**	**30**	**30**	**b**	**1**	**1**	**1**
x_p	5	2	6	y_p	1/6	1/15	1/5
η	8	4	9	θ	4/15	2/15	3/10
x_r	15	7	12	y_r	1/2	7/30	2/5
m	1.504	1.269	2.328	**m**	1.504	1.269	2.328
n	2.838	7.733	5.755	**n**	2.838	7.733	5.755

B	EXPERT 1	EXPERT 2	EXPERT 3		EXPERT 1	EXPERT 2	EXPERT 3
a	0	0	0	**a**	0	0	0
p	0.2	0.2	0.2	**p**	0.2	0.2	0.2
r	0.8	0.8	0.8	**r**	0.8	0.8	0.8
b	**100**	**100**	**100**	**b**	**1**	**1**	**1**
x_p	5	2	6	y_p	0.050	0.020	0.060
η	8	4	9	θ	0.080	0.040	0.090
x_r	15	7	12	y_r	0.150	0.070	0.120
m	1.592	1.288	2.360	**m**	1.592	1.288	2.360
n	13.402	29.491	26.015	**n**	13.402	29.491	26.015

3. PRIOR DIRICHLET PROCESS

Construction...

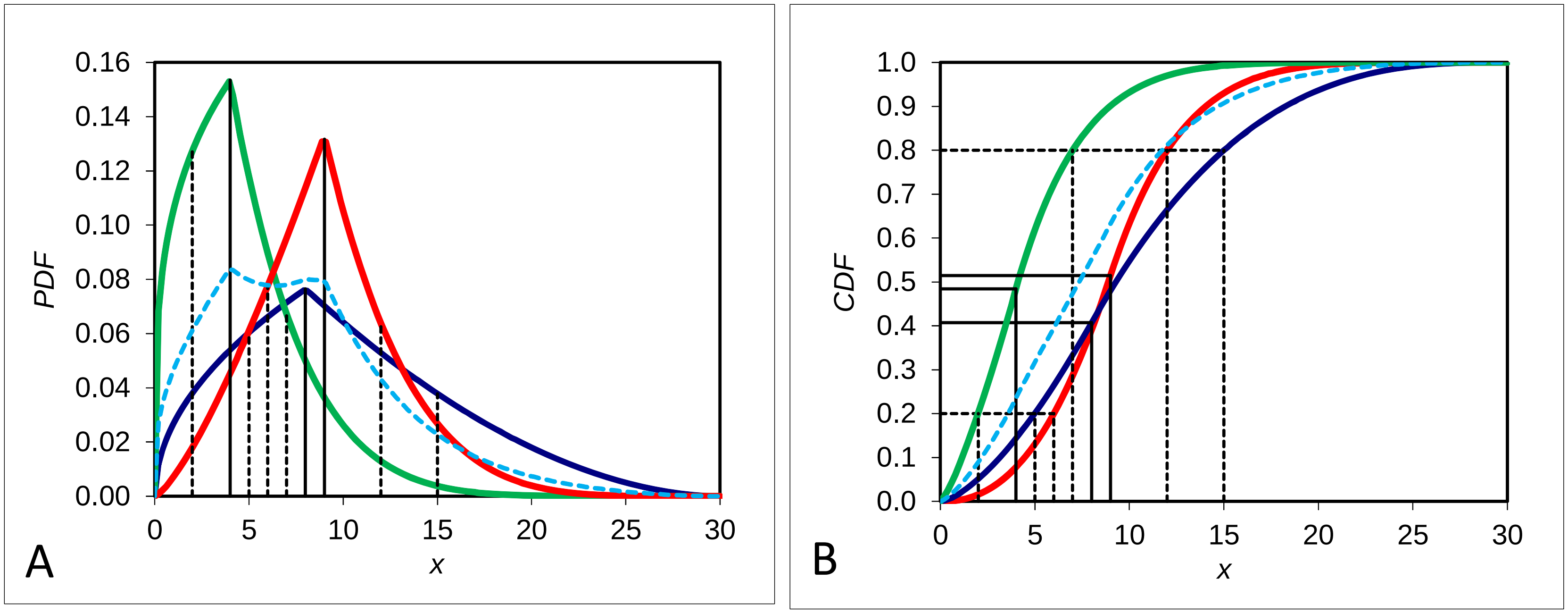


Figure 3. GTSP distribution for the expert data in Table 1. Expert 1's distribution in dark blue, Expert 2's distribution in green, Expert 3's distribution in red, equi-weight mixture distribution in light blue.

- **Step 2:** Fit **Generalized Trapezoidal** cdf $F(t|\Theta)$ to $F_d(t)$ (although not required for prior DP construction, but **provides parametric convenience**).

3. PRIOR DIRICHLET PROCESS

Construction...

- **The Generalized Trapezoidal cdf with support (a, b)** is given by:

$$F(x|\Theta) = \begin{cases} \frac{2\alpha(b-a)n_3}{2\alpha(\eta_1-a)n+(\alpha+1)(\eta_2-\eta_1)mn+2(b-\eta_2)m}\left(\frac{x-a}{\eta_1-a}\right)^m, & \text{for } a \leq x < \eta_1 \\ \frac{2\alpha(b-a)n_3+2(x-b)n_1n_3\left\{1+\frac{(\alpha-1)}{2}\frac{(2c-b-x)}{(c-b)}\right\}}{2\alpha(\eta_1-a)n+(\alpha+1)(\eta_2-\eta_1)mn+2(b-\eta_2)m}, & \text{for } \eta_1 \leq x < \eta_2 \\ 1-\frac{2(d-c)n_1}{2\alpha(\eta_1-a)n+(\alpha+1)(\eta_2-\eta_1)mn+2(b-\eta_2)m}\left(\frac{d-x}{d-\eta_2}\right)^n, & \text{for } \eta_2 \leq x < b. \end{cases}$$

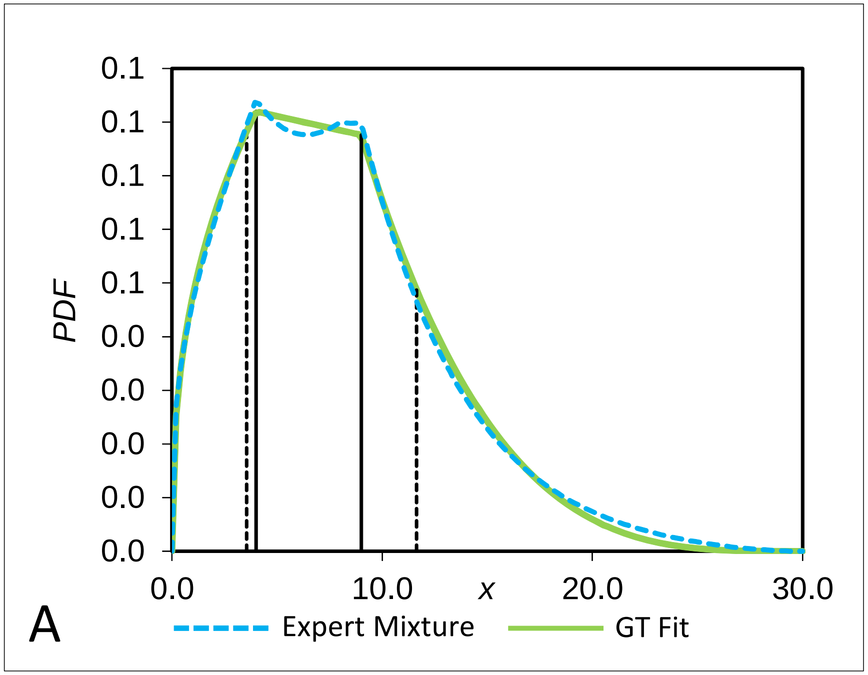
- Set $(a, b) = (0, 30)$, set $\eta_1 = 4$ (**the smallest elicited most likely estimate in Table 1**) and set $\eta_2 = 9$ (**the largest most likely estimate in Table 1**).

- Solve for **GT parameters α, m and n** of the using a least squares procedure between the equi-weight mixture cdf and the GT cdf, resulting in

$$\alpha = 1.056, m = 1.390, n = 4.464.$$

3. PRIOR DIRICHLET PROCESS

Construction...



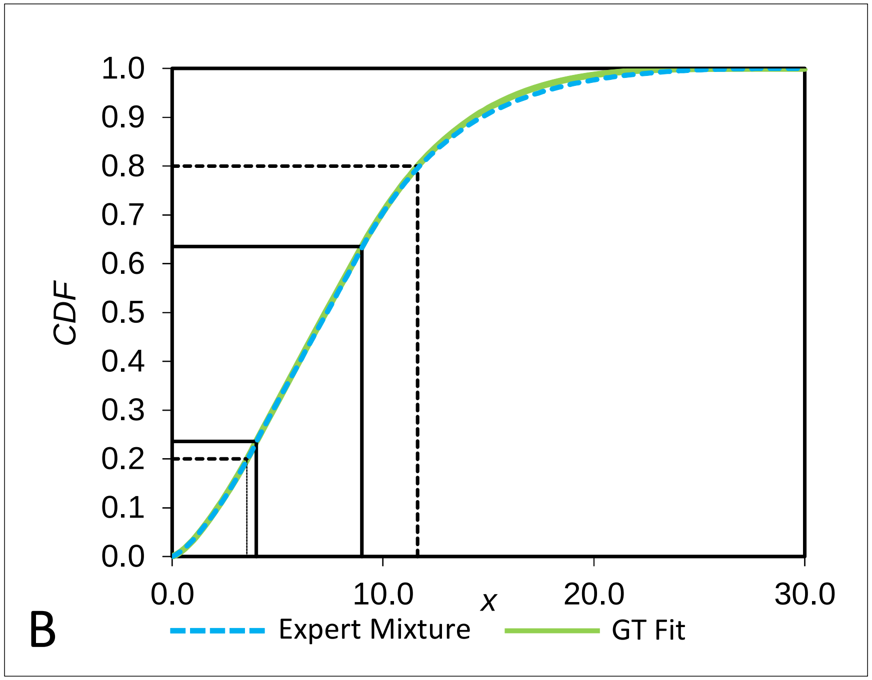


Figure 4. **Equi-weight mixture distribution (in light blue), GT fit to the mixture distribution (in light green). A: pdfs, B: cdfs.**

- **Step 3:** Encapsulate prior knowledge in the Dirichlet Process (DP) by setting:

$$\boldsymbol{\alpha}\{(0, x)\} = \boldsymbol{\alpha}(\mathbb{R}^{+}) \times F(x|\Theta).$$

3. PRIOR DIRICHLET PROCESS

Construction...

- This yields for the **Dirichlet Process**:

$$E[F(x)|\alpha(\,\cdot\,)] = F(x|\Theta), \text{i.e } \textbf{the fitted GT cdf} \tag{8}$$

$$V[F(x)|\alpha(\,\cdot\,)] = \frac{F(x|\Theta) \times \{1 - F(x|\Theta)\}}{\boldsymbol{\alpha}(\mathbb{R}^+) + 1}.$$

- Observe that $\boldsymbol{\alpha}(\mathbb{R}^+)$ is positive constant that **drives the variance in $\boldsymbol{F(x)}$**.

- **Step 4:** Evaluate $x^\bullet$ that maximizes

$$V\widehat{ar}[F(x)] = \frac{1}{E-1}\sum_{e=1}^{E}\{F_e(x) - F(x|\Theta)\}^2, \tag{9}$$

- **Step 5:** Solve $\boldsymbol{\alpha}(\mathbb{R}^+)$ from (9) by setting

$$V[F(x^\bullet)|\alpha(\,\cdot\,)] = \widehat{V}[F(x^\bullet)], \tag{10}$$

3. PRIOR DIRICHLET PROCESS

Construction...

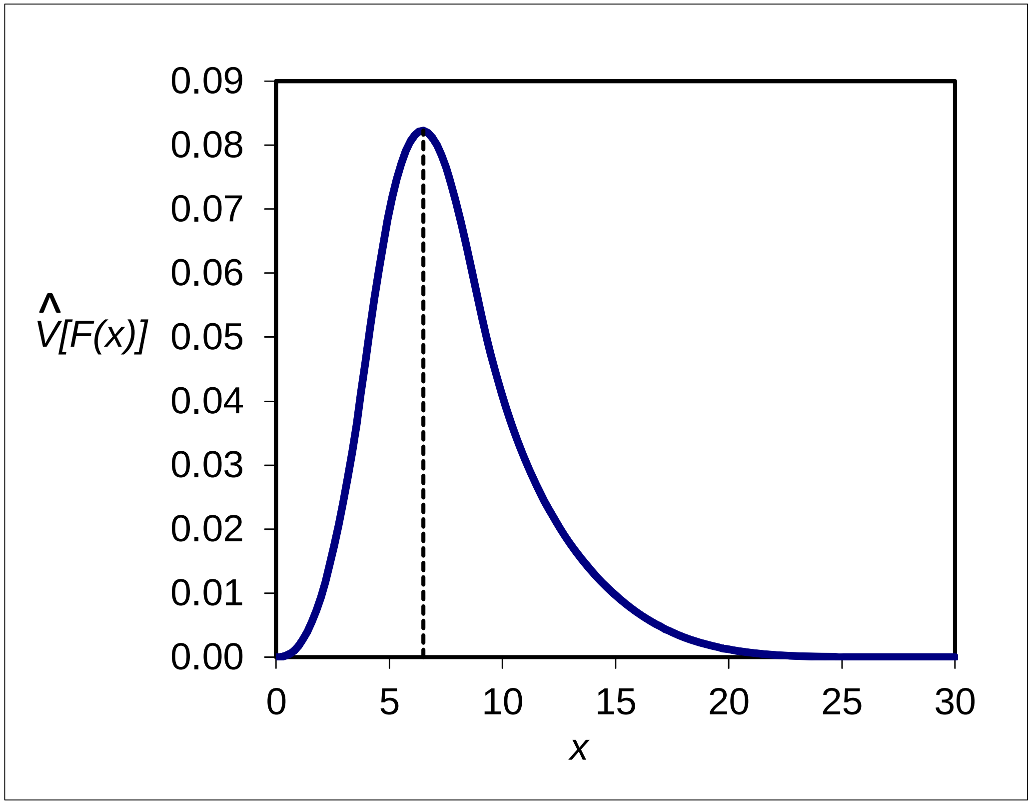


Figure 5. Plot of $\widehat{V}[F(x)]$ given by (9) for the example data in Table 1.

$$x^{\bullet} = 6.513 \text{ with } \widehat{V}[F(x^{\bullet})] = 0.0822 \text{ and } F(x^{\bullet}|\Theta) = 0.439 \Rightarrow \boldsymbol{\alpha}(\mathbb{R}^{+}) \approx 1.995.$$

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- **Failure Data:** $\{\boldsymbol{n_x}, \underline{x}\} \equiv \{x_{(1)}, \ldots x_{(n_x)}\}$ a sample of ordered fail. times x_j.

- **Ferguson (1973)'s** main theorem provides the form of $E[F(x)|\alpha(\,\cdot\,), \{n_x, \underline{x}\}]$, i.e. **the posterior expectation** for the lifetime distribution $\boldsymbol{F(x)}$ **given observed failure data** $\boldsymbol{\{n_x, \underline{x}\}}$.

- **Ferguson (1973)** demonstrated that

$$E[F(x)|\alpha(\,\cdot\,), \{n_x, \underline{x}\}\,] = \lambda_{n_x} F(x|\Theta) + (1-\lambda_{n_x})\widehat{F}_{n_x}(x|\{n_x, \underline{x}\}),$$

where

$$\lambda_{n_x} = \frac{\boldsymbol{\alpha}(\mathbb{R}^+)}{\boldsymbol{\alpha}(\mathbb{R}^+) + \boldsymbol{n_x}},$$

$$\widehat{F}_{n_x}(x|\{n_x, \underline{x}\}) = \frac{i}{\boldsymbol{n_x}} \text{ for } x_{(i)} \le x < x_{(i+1)}, i = 1, \ldots, n_x,$$

and $x_{(0)} \equiv 0$, $x_{(n_x+1)} \equiv \infty$.

$$\alpha(\mathbb{R}^+) \approx 1.995, n_x = 5, \Rightarrow \lambda_{n_x} \approx 0.285 \quad (11)$$
$$x_{(1)} = 4, x_{(2)} = 10, x_{(3)} = 11, x_{(4)} = 13, x_{(5)} = 15$$

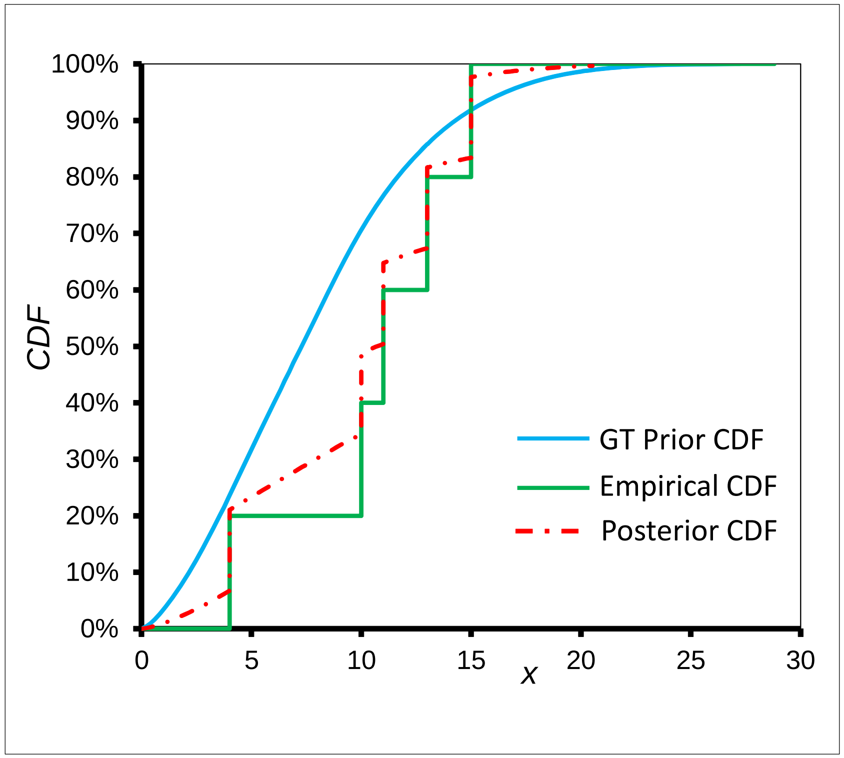

Figure 6. Comparison of prior GT cdf $F(x|\Theta)$, empirical cdf $\widehat{F}_{n_x}(x|\{n_x, \underline{x}\})$ and posterior cdf $E[F(x)|\alpha(\cdot), \{n_x, \underline{x}\}]$ given failure data (11).

- **Maintenance Data:**

$$\{n_c, (\underline{\gamma}, \underline{c})\} \equiv [\{\gamma_1, c_{(1)}\}, \ldots \{\gamma_2, c_{(n_c)}\}]$$

where $\{\gamma_j, c_{(j)}\}$ indicates that t**he component was removed from service $\boldsymbol{\gamma_j}$ times** at **censor time $\boldsymbol{c_{(j)}}$** to **be preventively maintained**.

- **Join the failure data $\{\boldsymbol{n_x}, \underline{\boldsymbol{x}}\}$ with maintenance data $\{\boldsymbol{n_c}, (\underline{\boldsymbol{\gamma}}, \underline{\boldsymbol{c}})\}$:**

$$\{n_z, (\underline{\delta}, \underline{z})\} = \{(\delta_1, z_{(1)}), \ldots, (\delta_{m_z}, z_{(m_z)})\},$$

$$m_z = n_x + n_c, \; \boldsymbol{n_z} = \boldsymbol{n_x} + \sum_{j=1}^{n_c} \gamma_i,$$

$$\delta_j = \begin{cases} 1, & z_{(j)} \in \{x_{(1)}, \ldots x_{(n_x)}\}, \\ \gamma_j, & \{\gamma_j, z_{(j)}\} \in [\{\gamma_1, c_{(1)}\}, \ldots \{\gamma_2, c_{(n_c)}\}]. \end{cases}$$

4. BAYESIAN UPDATING

... and Maintenance Data

- **Susarla and Van Ryzin (1976)** derived the following **Bayes estimator for the component survival function** when $c_{(k)} \leq t < c_{(k+1)}, k = 0, \ldots, n_c,$ $c_{(0)} \equiv 0, c_{(n_c+1)} \equiv \infty:$

$$\widehat{S}(x \,|\, \Psi\} = \frac{\boldsymbol{\alpha\{(x, \infty)\}} + n^+(t)}{\boldsymbol{\alpha(\mathbb{R}^+)} + \boldsymbol{n_z}} \times \prod_{j=1}^{k} \frac{\boldsymbol{\alpha\{[c_{(j)}, \infty)\}} + n(c_{(j)})}{\boldsymbol{\alpha\{[c_{(j)}, \infty)\}} + n(c_{(j)}) - \gamma_j}$$

where $\Psi = [\alpha(\,\cdot\,), (n_x, \underline{x}), \{n_c, (\underline{\gamma}, \underline{c})\}]$ and **$\boldsymbol{\alpha(\,\cdot\,)}$ is the parameter measure of a Dirichlet process**, by convention $\prod_{j=1}^{0} \{\,\cdot\,\} \equiv 0, n_z\,, \delta_j, \gamma_j$ defined as before, and finally

$$n^+(x) = \sum_{\{i: z_{(i)} > x\}} \delta_i, \text{ and } n(x) = \sum_{\{i: z_{(i)} \geq x\}} \delta_i.$$

4. BAYESIAN UPDATING

... and Maintenance Data

- Setting $\boldsymbol{\alpha}\{(\boldsymbol{x},\infty)\} = \boldsymbol{\alpha}(\mathbb{R}^+) \times S(x|\Theta))$:

$$\widehat{S}(x\,|\,\Psi) = \left\{\lambda_{n_z} S(x|\Theta) + (1-\lambda_{n_z})\widehat{\boldsymbol{S}}_{\boldsymbol{n_z}}[\boldsymbol{x}|\{\boldsymbol{n_z},(\underline{\boldsymbol{\delta}},\underline{\boldsymbol{z}})\}]\right\} \times$$
$$\prod_{j=1}^{k} \frac{\boldsymbol{\alpha}(\mathbb{R}^+) \times S(c_{(j)}|\Theta) + n(c_{(j)})}{\boldsymbol{\alpha}(\mathbb{R}^+) \times S(c_{(j)}|\Theta) + n(c_{(j)}) - \gamma_j}.$$

$$\lambda_{n_z} = \frac{\boldsymbol{\alpha}(\mathbb{R}^+)}{\boldsymbol{\alpha}(\mathbb{R}^+) + \boldsymbol{n_z}}, S(x|\Theta) = 1 - F(x|\Theta),$$

$$\widehat{S}_{\boldsymbol{n_z}}[\boldsymbol{x}|\{\boldsymbol{n_z},(\underline{\boldsymbol{\delta}},\underline{\boldsymbol{z}})\}] = \frac{n^+(x)}{\boldsymbol{n_z}},$$

- **Example:** $\{n_c,(\underline{\gamma},\underline{c})\} \equiv \{(4,3),(3,6),(2,9),(1,12)\} \Rightarrow n_c = 10$, $\alpha(\mathbb{R}^+) \approx 1.995$, Failure data $(11) \Rightarrow n_z = 15 \;\Rightarrow\; \lambda_{n_z} \approx 0.117$.

4. BAYESIAN UPDATING

… and Maintenance Data

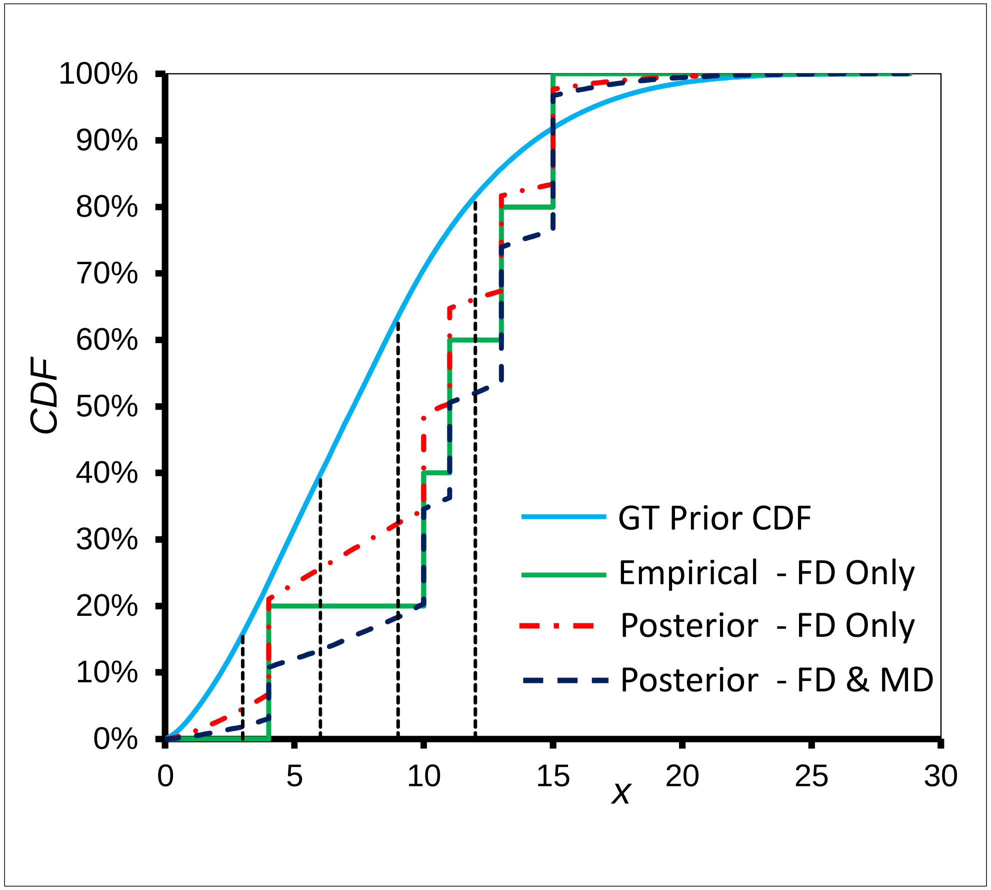

Figure 7. Comparison of prior GT cdf $F(x|\Theta)$, empirical cdf $\widehat{F}_{n_x}(x|\{n_x, \underline{x}\})$, posterior cdf $E[F(t)|\alpha(\cdot), \{n_x, \underline{x}\}]$ given failure data (11) and posterior cdf $\widehat{F}(t\,|\,\Psi) = 1 - \widehat{S}(t\,|\,\Psi)$ given failure data (11) and maintenance data.

- **Susarla and Van Ryzin (1976)** assumed random observations $\boldsymbol{Z_i = \min(X_i, C_i)}$, where the X_i random failure times are $i.i.d$, and the C_i's are random censoring times also independent from the X_i's.

- **The $\boldsymbol{C_i}$ random variables** are assumed to **be mutually independent**, but do not have to be identically distributed and **could be degenerate implying fixed maintenance times**.

- **In case of no censoring $\boldsymbol{n_z = n_x}$**, $\widehat{S}_{n_z}[x|\{n_z, (\underline{\delta}, \underline{z})\}]$ reduces to **the empirical survival function** given failure data $\{n_x, \underline{x}\}$, and **the product term reduces to the value 1** since $k = 0$ in the no censoring case. Hence, the **Susarla and Van Ryzin (1976) formula reduces to Ferguson (1973)'s**.

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5. MAINTENANCE OPTIMIZATION
Block Replacement Model ...

- **A basic model** within the context of **maintenance optimization** is **the block replacement model.** For an extensive discussion of this model see **Mazzuchi and Soyer (1996)**.

- In the block replacement model, **a single maintenance activity is carried out at a pre-specified age $\boldsymbol{x}$** of the component.

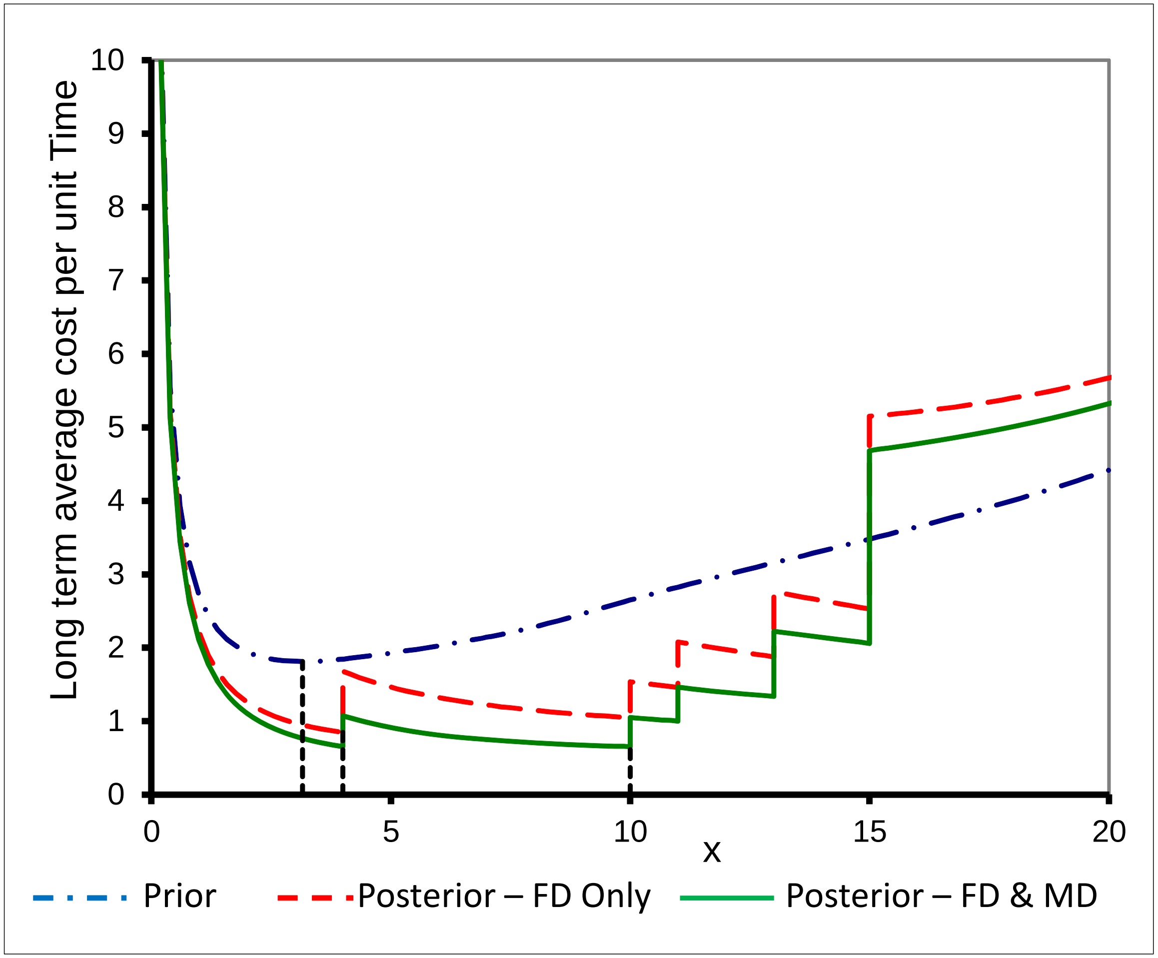
- One obtains for **the long term average cost per unit time:**

$$g(x) = \frac{\mathcal{K}_p + \mathcal{K}_f \times \Lambda(x)}{x},$$

where $\Lambda(x) \equiv$ the expected number of failures during the maintenance cycle x, $\boldsymbol{\mathcal{K}_f}$ **are the expected failure cost** and $\boldsymbol{\mathcal{K}_p}$ **are the preventive maintenance cost**. As $\mathcal{K}_f$ is unplanned it is assumed that $\mathcal{K}_f > \mathcal{K}_p$.

5. MAINTENANCE OPTIMIZATION

Block Replacement Model ...



$\mathcal{K}_f = 20$ and $\mathcal{K}_p = 2$, i.e. a failure is **ten times more costly** than a preventative maintenance action.

OUTLINE

1. INTRODUCTION
2. THREE POINT ELICITATION
3. PRIOR DIRICHLET PROCESS CONSTRUCTION
4. BAYESIAN UPDATING USING FAILURE AND MAINTENANCE DATA
5. MAINTENANCE OPTIMIZATION
6. **SELECTED REFERENCES**

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